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1987 J. Phys. A: Math. Gen. 20 5015

(http://iopscience.iop.org/0305-4470/20/14/037)

J. Phys. A: Math. Gen. 20 (1987) 5015-5021. Printed in the UK

COMMENT

New perspective on the U(n) Wigner–Racah calculus: III. Applications to U(2) and U(3)

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Received 6 October 1986, in final form 13 March 1987

Abstract. Using the framework introduced in parts I and II, we derive an interesting relationship between U(3) symmetric tensors and U(2) Racah coefficients. Closed formulae for generic U(2) Wigner and Racah coefficients are given in terms of alternative parametrisations for such coefficients.

1. U(3) symmetric tensors and U(2) Racah coefficients

It has been shown in part II of this series (Le Blanc and Hecht 1987) that in certain cases the elementary reduced Wigner coefficients for U(n) are given by the product of a U($n-1$) Racah coefficient times some dimensional factors and normalisation factors (K matrices) arising from a vector coherent state theory (Rowe 1984, Rowe *et al* 1985, Rowe and Carvalho 1986, Le Blanc and Rowe 1985a, b, Hecht and Elliott 1985, Hecht *et al* 1987a, b). In particular, we have obtained

$$\left\langle \begin{matrix}\{m_{13}m_{23}0\} \\ \{m_{12}m_{22}\}\end{matrix} ; \begin{matrix}\{100\} \\ \{10\}\end{matrix} \middle\| \begin{matrix}\{m'_{13}m'_{23}0\} \\ \{m'_{12}m'_{22}\}\end{matrix} \right\rangle$$
$$= (-1)^{\phi(\{m'\}_3)-\phi\{m\}_3)+\phi(\{m\}_2)-\phi(\{m'\}_2)-1/2}$$
$$\times \left(\frac{\dim\{m'_{13}m'_{23}\}_2 \dim\{m_{12}m_{22}\}_2}{\dim\{m_{13}m_{23}\}_2 \dim\{m'_{12}m'_{22}\}_2}\right)^{1/2} K\begin{pmatrix}\{m_{13}m_{23}0\} \\ \{m_{12}m_{22}\}\end{pmatrix} K\begin{pmatrix}\{m'_{13}m'_{23}0\} \\ \{m'_{12}m'_{22}\}\end{pmatrix}^{-1}$$
$$\times U(\{w0\}\{m_{12}m_{22}\}\{m'_{13}m'_{23}\}\{10\}; \{m_{13}m_{23}\}\{m'_{12}m'_{22}\}) \tag{1.1}$$

for one of the $U(3)\supset U(2)$ fundamental reduced Wigner coefficients in the usual Gel'fand notation. In equation (1.1), the U coefficient is a U(2) Racah coefficient, $\phi(\{m\}_n)$ is a U(n) phase factor given by

$$\begin{aligned}\phi(\{m\}_n) &= \tfrac{1}{2}\sum_{i<j}^{n} (m_{in}-m_{jn}) \\ &= \tfrac{1}{2}\sum_{i=1}^{n} (n+1-2i)m_{in}\end{aligned} \tag{1.2}$$

(see Hecht *et al* 1987b) and

$$w = m_{13}+m_{23}-m_{12}-m_{22}. \tag{1.3}$$

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0305-4470/87/145015+07$02.50

Also

$$\dim(\{f_1 f_2\}_2) = f_1 - f_2 + 1$$

is the dimension of the U(2) unirrep $\{f_1 f_2\}_2$ and $K(\ )$ is a normalisation factor arising from the above-mentioned vector coherent state theory. For generic U(3) unirreps, it is given by (Hecht *et al* 1987b)

$$K^2\begin{pmatrix}\{m_{13}m_{23}m_{33}\}\\ \{m_{12}m_{22}\}\end{pmatrix} = \prod_{i=1}^{2} \frac{(p_{i3}-p_{33}-1)!}{(p_{i2}-p_{33})!} \tag{1.4a}$$

in terms of the well known hook quantity

$$p_{ij} = m_{ij} + j - i \tag{1.4b}$$

of the representation theory of the symmetric group (Robinson 1961).

Generalising the steps leading to equation (1.1) (see Le Blanc and Hecht 1987), we readily find that

$$\left\langle \begin{matrix}\{m_{13}m_{23}0\}\\ \{m_{12}m_{22}\}\end{matrix};\ \begin{matrix}\{\lambda 00\}\\ \{\lambda 0\}\end{matrix} \right\Vert \left. \begin{matrix}\{m'_{13}m'_{23}0\}\\ \{m'_{12}m'_{22}\}\end{matrix}\right\rangle$$
$$= (-1)^{\phi(\{m'\}_3)-\phi(\{m\}_3)+\phi(\{m\}_2)-\phi(\{m'\}_2)-\lambda/2}$$
$$\times\left(\frac{\dim\{m'_{13}m'_{23}\}_2 \dim\{m_{12}m_{22}\}_2}{\dim\{m_{13}m_{23}\}_2 \dim\{m'_{12}m'_{22}\}_2}\right)^{1/2} K\begin{pmatrix}\{m_{13}m_{23}0\}\\ \{m_{12}m_{22}\}\end{pmatrix} K\begin{pmatrix}\{m'_{13}m'_{23}0\}\\ \{m'_{12}m'_{22}\}\end{pmatrix}^{-1}$$
$$\times U(\{w0\}\{m_{12}m_{22}\}\{m'_{13}m'_{23}\}\{\lambda 0\}; \{m_{13}m_{23}\}\{m'_{12}m'_{22}\}). \tag{1.5}$$

The only real unknown in equation (1.5) is the U(2) Racah coefficient itself for which we now seek a closed analytical expression.

When (1.5) is introduced into the defining equation for the multiplicity-free U(3) Racah coefficient

$$U(\{m_{13}m_{23}0\}\{\lambda-1,0,0\}\{m'_{13}m'_{23}0\}\{100\}; \{m''_{13}m''_{23}0\}\{\lambda 00\})$$

(Hecht 1965), we find after simplification (Le Blanc 1986) that the equation reduces to

$$U(\{w0\}\{m_{12}m_{22}\}\{m'_{13}m'_{23}\}\{\lambda 0\}; \{m_{13}m_{23}\}\{m'_{12}m'_{22}\})$$
$$\times U(\{m_{13}m_{23}\}\{\lambda-1,0\}\{m'_{13}m'_{23}\}\{10\}; \{m''_{13}m''_{23}\}\{\lambda 0\})$$
$$= \sum_{\{m''\}_2} U(\{w0\}\{m_{12}m_{22}\}\{m''_{13}m''_{23}\}\{\lambda-10\}; \{m_{13}m_{23}\}\{m''_{12}m''_{22}\})$$
$$\times U(\{w0\}\{m''_{12}m''_{22}\}\{m'_{13}m'_{23}\}\{10\}; \{m''_{13}m''_{23}\}\{m'_{12}m'_{22}\})$$
$$\times U(\{m_{12}m_{22}\}\{\lambda-1,0\}\{m'_{12}m'_{22}\}\{10\}; \{m''_{12}m''_{22}\}\{\lambda 0\}). \tag{1.6}$$

Thus, a specific recursion relation between U(3) : U(2) reduced Wigner coefficients and U(3) Racah coefficients simplifies to the Biedenharn–Elliott–Racah fundamental identity for U(2) Racah coefficients (Biedenharn 1953, Elliott 1953).

Introducing the well known expressions for the fundamental U(2) Racah coefficients (see the appendix of Le Blanc and Hecht (1987) for a useful parametrisation of such coefficients), we find the following two-term recursion formula for a generic U(2) Racah coefficient:

$$U(\{w0\}\{m_{12}m_{22}\}\{m_{13}+\lambda-l,\ m_{23}+l\}\{\lambda 0\};\ \{m_{13}m_{23}\}\{m_{12}+\lambda-k,\ m_{22}+k\})$$
$$=\left(\frac{(\lambda-k)(p_{12}-p_{22}+\lambda-k)(p_{13}-p_{12}+k-l)(p_{22}-p_{23}+k-l+1)}{l(p_{13}-p_{23}-l)(p_{12}-p_{22}+\lambda-2k)(p_{22}-p_{12}-\lambda+2k+1)}\right)^{1/2}$$
$$\times U(\{w0\}\{m_{12}m_{22}\}\{m_{13}+k-l,\ m_{23}+l-1\}$$
$$\times\{\lambda-1,0\};\ \{m_{13}m_{23}\}\{m_{12}+\lambda-k-1,\ m_{22}+k\})$$
$$+\left(\frac{k(p_{12}-p_{22}-k)(p_{13}-p_{22}+\lambda-k-l)(p_{12}-p_{23}+\lambda-k-l+1)}{l(p_{13}-p_{23}-l)(p_{12}-p_{22}+\lambda-2k)(p_{12}-p_{22}+\lambda-2k+1)}\right)^{1/2}$$
$$\times U(\{w0\}\{m_{12}m_{22}\}\{m_{13}+k-l,\ m_{23}+l-1\}$$
$$\times\{\lambda-1,0\};\{m_{13}m_{23}\}\{m_{12}+\lambda-k,\ m_{22}+k-1\}). \tag{1.7}$$

The reader is invited to compare our parametrisation with the one used by Biedenharn and Louck (1981b, ch 4).

With, as a starting point, the easily derived value

$$U(\{w0\}\{m_{12}m_{22}\}\{m_{13}+\lambda,\ m_{23}\}\{\lambda 0\};\ \{m_{13}m_{23}\}\{m_{12}+\lambda-k,\ m_{22}+k\})$$
$$=(-1)^k\left[\binom{\lambda}{k}\frac{(p_{13}-p_{22})_{\lambda-k}(p_{13}-p_{12})_k}{(p_{13}-p_{23}+1)_\lambda}\times\frac{(p_{12}-p_{23}+1)_{\lambda-k}(p_{22}-p_{23}+1)_k}{(p_{12}-p_{22}-k)_{\lambda-k}(p_{22}-p_{12}-\lambda+k)_k}\right]^{1/2} \tag{1.8}$$

for the stretched ($l=0$) case (Brink and Satchler 1968) where $(x)_a=x(x+1)\dots(x+a-1)$ is a raising factorial (Pochhammer symbol), we easily obtain by recursion on l (with, for now, $l\leqslant k$ and $l\leqslant\lambda-k$)

$$U(\{w0\}\{m_{12}m_{22}\}\{m_{13}+\lambda-l,\ m_{23}+l\}\{\lambda 0\};\ \{m_{13}m_{23}\}\{m_{12}+\lambda-k,\ m_{22}+k\})$$
$$=(-1)^k\left(\frac{l!(\lambda-l)!}{k!(\lambda-k)!}\,\frac{(p_{13}-p_{22})_{\lambda-k-l}(p_{13}-p_{12})_{k-l}}{(p_{13}-p_{23}+1)_{\lambda-l}(p_{23}-p_{13}+1)_l}\times\frac{(p_{12}-p_{23}+1)_{\lambda-k-l}(p_{22}-p_{23}+1)_{k-l}}{(p_{12}-p_{22}-k)_{\lambda-k}(p_{22}-p_{12}-\lambda+k)_k}\right)^{1/2}$$
$$\times\sum_{i=0}^{l}\left\{\binom{k}{i}\binom{\lambda-k}{l-i}\times\frac{(p_{13}-p_{12}+k-l)_{l-i}(p_{22}-p_{23}+1+k-l)_{l-i}(p_{22}-p_{12}-\lambda+k)_{l-i}}{(p_{12}-p_{22}+1+\lambda-l-2k+2i)_{l-i}}\right.$$
$$\left.\times\frac{(p_{13}-p_{22}+\lambda-k-l)_i(p_{12}-p_{23}+1+\lambda-k-l)_i(p_{12}-p_{22}-k)_i}{(p_{22}-p_{12}+1-\lambda+l+2k-2i)_i}\right\}. \tag{1.9a}$$

When due consideration is taken of the respective values of λ, l and k, we obtain, with

$$i_o=\max(0,\ l+k-\lambda)\qquad i_1=\min(l,\ k)$$

the slightly modified form

$$
\begin{aligned}
&U(\{w0\}\{m_{12}m_{22}\}\{m_{13}+\lambda-l,\, m_{23}+l\}\{\lambda 0\};\, \{m_{13}m_{23}\}\{m_{12}+\lambda-k,\, m_{22}+k\})\\
&\quad=(-1)^{k+l-i_1}\left(\frac{l!(\lambda-l)!}{k!(\lambda-k)!}\,\frac{(p_{13}-p_{22})_{\lambda-k-l+i_o}(p_{13}-p_{12})_{k-i_1}}{(p_{13}-p_{23}+1)_{\lambda-l}(p_{23}-p_{13}+1)_l}\right.\\
&\qquad\left.\times\frac{(p_{12}-p_{23}+1)_{\lambda-k-l+i_o}(p_{22}-p_{23}+1)_{k-i_1}}{(p_{12}-p_{22}-k)_{\lambda-k}(p_{22}-p_{12}-\lambda+k)_k}\right)^{1/2}\\
&\qquad\times[(p_{13}-p_{22}+\lambda-k-l)_{i_o}(p_{13}-p_{12}+k-l)_{l-i_1}(p_{12}-p_{23}+1+\lambda-k-l)_{i_o}\\
&\qquad\times(p_{22}-p_{23}+1+k-l)_{l-i_1}]^{1/2}\sum_{i=i_o}^{i_1}\left\{\binom{k}{i}\binom{\lambda-k}{l-i}\right.\\
&\qquad\times\frac{(p_{13}-p_{12}+k-i_1)_{i_1-i}(p_{22}-p_{23}+1+k-i_1)_{i_1-i}(p_{22}-p_{12}-\lambda+k)_{l-i}}{(p_{12}-p_{22}+1+\lambda-l-2k+2i)_{l-i}}\\
&\qquad\left.\times\frac{(p_{13}-p_{22}+\lambda-k-l+i_o)_{i-i_o}(p_{12}-p_{23}+1+\lambda-k-l+i_o)_{i-i_o}(p_{12}-p_{22}-k)_i}{(p_{22}-p_{12}+1-\lambda+l+2k-2i)_i}\right\}.
\end{aligned}
\tag{1.9b}
$$

Our phase convention agrees with the ones of Condon-Shortley and Biedenharn-Louck.

Setting, without loss of generality, $m_{22}=0$, using the usual angular momentum notation

$$
\begin{aligned}
&2j_1=w \qquad 2j_2=m_{12}-m_{22}=m_{12} \qquad 2j_3=\lambda\\
&2j_{12}=m_{13}-m_{23} \qquad 2j_{23}=m_{12}-m_{22}+\lambda-2k=m_{12}+\lambda-2k\\
&2j=m_{13}-m_{23}+\lambda-2l
\end{aligned}
\tag{1.10}
$$

and recalling equation (1.3) (which reduces the number of independent parameters to six, i.e. m_{12}, m_{13}, w, λ, k and l), we conclude that equation (1.9) is a closed analytical expression for the generic SU(2) Racah coefficients $U(j_1, j_2, j, j_3; j_{12}, j_{23})$.

As discussed in Biedenharn and Louck (1981a, b), it is only when expressed in terms of the hooks p_{ij} that expressions for U(n) Wigner-Racah coefficients can be given relatively simple forms such as (1.8) and (1.9) and it is for this reason that we have chosen to use the hook notation. It is then relatively straightforward to study the symmetries of the various coefficients invoking concepts such as the permutational symmetries of the hooks, of the shifts, etc. The main difference between our parametrisation and the one of Biedenharn and Louck (1981a, b) is that we have used from the outset the relationship between U(3) : U(2) reduced Wigner coefficients and U(2) Racah coefficients to introduce the hooks p_{13}, p_{23}, p_{12}, p_{22} while Biedenharn and Louck enlarged on the pattern calculus rules for the computation of U(2) : U(1) Wigner coefficients to introduce the hooks p_{12}, p_{22}, p_{11} and p_{21}. This last quantity is ill defined in terms of standard Gel'fand patterns but turns out to be an extremely useful concept for an extended pattern calculus.

Although not obvious, the terms under the summation sign in (1.9) reduce to an irreducible polynomial, i.e. when divided by a common denominator the summation reduces to a simpler polynomial with no such denominator and the resulting quantity is an integer. Unfortunately, and as mentioned by Biedenharn and Louck (1981a, b), the recursion formula (1.7) does not naturally bring forward the polynomial and some supplementary concepts must be called upon in order to offer a simpler derivation of the polynomial.

2. SU(2) Wigner coefficients

In part II (Le Blanc and Hecht 1987) of this series, it was demonstrated that a U(n) Wigner-Racah calculus could be implemented in a minimal U($n-1$)$\times$U(n) complementary space of double Bargmann-Gel'fand polynomials. It was also mentioned there that an unambiguous set of rules for the construction of Biedenharn-Louck-Bargmann (see, e.g., Louck 1970, Le Blanc and Rowe 1986a, b) U(n) shift tensors were still lacking except for the case of multiplicity-free couplings.

Fortunately, the U(2) case belongs entirely to the case of simple reducibility. Furthermore, the U(1)$\times$U(2) Bargmann space then collapses to the well known Schwinger (1965) representation space of functions in two Bargmann complex variables (g_+, g_-). The upper U(1) (Abelian) group of transformations then simplifies to a simple boson conservation law and the construction of the Bargmann shift tensors is then unambiguous.

The purpose of this section is thus to exploit the simple structure of the U(2) Bargmann shift tensors to derive closed expressions for the SU(2) Wigner coefficients. Our expressions will be seen to be similar, except for the very explicit parametrisation, to the 'reduced expression' of Clebsch-Gordan coefficients given by Sato and Kaguei (1972). The Sato and Kaguei reduced expression for SU(2) Wigner coefficients, coined 'symbolic expression' by Biedenharn and Louck (1981b), was obtained by inspection of the well known Racah formulae for these coefficients. It can be shown (Bargmann 1962, Le Blanc 1986) that such an expression is the outcome of some simple recursion formulae for the Wigner coefficients. We use an unambiguous parametrisation of the Wigner coefficient (related in part to the upper pattern symbolism of Biedenharn and Louck but also to Bargmann's ingenious notation) and consequently give simple expressions for the coefficients which directly reflect the 72 Regge (1958) symmetries known to apply to these coefficients.

In the spirit of Bargmann's analysis (1962) of the rotation group in a Hilbert space of polynomials of complex (Bargmann) variables, we define the following useful quantities:

$$J = j_1 + j_2 + j_3$$
$$k_i = J - 2j_i \qquad s_i = j_i + m_i \qquad d_i = j_i - m_i \qquad 1 \leq i \leq 3. \tag{2.1}$$

Bargmann polynomials carrying components of an irreducible representation of angular momentum j are given by

$$\langle g|jm\rangle = \frac{g_+^{j+m} g_-^{j-m}}{[(j+m)!(j-m)!]^{1/2}} = \frac{g_+^s g_-^d}{\sqrt{s!d!}}. \tag{2.2}$$

Shift tensors for SU(2) can be constructed in terms of the two Bargmann variables g_+, g_- and their Hermitian conjugates (∂_+, ∂_-). We set the normalisation of these tensors by writing down explicitly their highest weight ($j_2 = m_2$) component:

$$T^{k_3}_{j_2, m_2=j_2}(g) = (-1)^{k_3} g_+^{2j_2-k_3} (\partial_-)^{k_3} = (-1)^{k_3} g_+^{k_1} (\partial_-)^{k_3}. \tag{2.3}$$

The irreducible tensor (2.3) will map the SU(2) unirrep j_1 to

$$T^{k_3}_{j_2} : j_1 \to j_1 + j_2 - k_3 = j_3. \tag{2.4}$$

Iteration of a two-term recursion formula on k_3 (see equation (2.3)) yields the following expression:

$$\langle j_1 m_1; j_2 m_2 | j_3 = j_1 + j_2 - k_3, m_3\rangle$$
$$= \left(\frac{(2j_3+1)}{(J+1)!}\frac{k_1!k_2!}{k_3!}\frac{s_3!d_3!}{s_1!d_1!s_2!d_2!}\right)^{1/2} F_{k_3}(s_1, d_2; s_2, d_1) \tag{2.5}$$

for SU(2) Wigner coefficients where the functional $F_{k_3}(\ )$ has an expansion reminiscent of the binomial expansion:

$$F_{k_3}(s_1, d_2; s_2, d_1) = \sum_{i=0}^{k_3} (-1)^i \binom{k_3}{i} [s_1]_{k_3-i}[d_2]_{k_3-i}[s_2]_i[d_1]_i. \tag{2.6}$$

In (2.6), the expression $[x]_i$ is a lowering factorial defined by $[x]_i = (x)(x-1)(x-2)\ldots(x-i+1)$ with $[x]_o = 1$.

An alternative recursive computation on k_1 (see equation (2.3)) yields

$$\langle j_1 m_1; j_2 m_2; | j_3, -m_3\rangle$$
$$= (-1)^{k_1 - s_2}\left(\frac{(2j_3+1)}{(J+1)!}\frac{k_2!k_3!}{k_1!}\frac{s_1!d_1!}{s_2!d_2!s_3!d_3!}\right)^{1/2} F_{k_1}(s_2, d_3; s_3, d_2). \tag{2.7}$$

Equations (2.5) and (2.7) are equivalent, except for the very explicit parametrisation used here, to the 'symbolic' expression of Sato and Kageui (1972) which they wrote down by inspection of the van der Waerden expression (1932). We have thus shown that these symbolic expressions have their origin in the iteration of very specific recursion formulae (compare with Bargmann 1962, § 3h), thus invalidating the assertion of Biedenharn and Louck (1981b) that 'there is no (known) theory for such symbolic techniques'.

The functional F_{k_3} is well defined for any (positive integer) value of the parameter k_3. It is trivial to prove the following symmetry properties of F which are fundamental to the verification of the Regge symmetries (Regge 1958, Le Blanc 1986):

$$F_{k_3}(s_1, d_2; s_2, d_1) = F_{k_3}(d_2, s_1; s_2, d_1) \tag{2.8a}$$
$$= F_{k_3}(s_1, d_2; d_1, s_2) \tag{2.8b}$$
$$= (-1)^{k_3} F_{k_3}(s_2, d_1; s_1, d_2). \tag{2.8c}$$

Another interesting property needed for the recursive proof leading to (2.5) is

$$F_{k_3}(s_1, d_2; s_2, d_1) = s_1 d_2 F_{k_3-1}(s_1 - 1, d_2 - 1; s_2, d_1)$$
$$- s_2 d_1 F_{k_3-1}(s_1, d_2; s_2 - 1, d_1 - 1). \tag{2.9}$$

Equation (2.5) enables one to easily construct condensed Condon–Shortley type tables for generic SU(2) Wigner coefficients. Rewriting all the quantities in (2.5) in terms of the unique parameter k_3, with $m = m_1 + m_2$ and

$$\left(\frac{N(k_3; j_1 m_1 j_2 m_2)}{D(k_3; j_1 m_1 j_2 m_2)}\right)^{1/2} \equiv \left(\frac{N(k_3)}{D(k_3)}\right)^{1/2}$$
$$= \left(\frac{(2j_1+2j_2+1-2k_3)}{(2j_1+2j_2+1-k_3)!}\frac{(2j_1-k_3)!(2j_2-k_3)!}{k_3!}\right.$$
$$\left.\times\frac{(j_1+j_2+m-k_3)!(j_1+j_2-m-k_3)!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\right)^{1/2} \tag{2.10}$$

we have

$$\langle j_1 m_1, j_2 m_2 | j_1 + j_2, m \rangle = \left(\frac{N(0)}{D(0)}\right)^{1/2}$$

$$\langle j_1 m_1, j_2 m_2 | j_1 + j_2 - 1, m \rangle = \left(\frac{N(1)}{D(1)}\right)^{1/2} [(j_1 + m_1)(j_2 - m_2) - (j_1 - m_1)(j_2 + m_2)]$$

$$\langle j_1 m_1, j_2 m_2 | j_1 + j_2 - 2, m \rangle = \left(\frac{N(2)}{D(2)}\right)^{1/2} [(j_1 + m_1)(j_1 + m_1 - 1)(j_2 - m_2)(j_2 - m_2 - 1)$$
$$- 2(j_1 + m_1)(j_2 - m_2)(j_1 - m_1)(j_2 + m_2) + (j_1 - m_1)(j_1 - m_1 - 1)(j_2 + m_2)(j_2 + m_2 - 1)]$$
$$\vdots$$
$$\langle j_1 m_1, j_2 m_2 | j_1 + j_2 - k_3, m \rangle = \left(\frac{N(k_3)}{D(k_3)}\right)^{1/2} F_{k_3}((j_1 + m_1)(j_2 - m_2); (j_2 + m_2)(j_1 - m_1))$$
$$\vdots \qquad (2.11)$$

which reproduces the standard tables when substituting for j_2 and m_2, but note that j_2 can now be arbitrarily large.

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